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SOME INVERSE PROBLEMS FOR FLEXIBLE PLATES

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We investigate the problems of finding the optimal loads acting on a plate, which insure the best root-mean-square (RMS) approximation to a given distribution of bending and torsional moments, or of the displacements. We study the problems of existence and uniqueness of the optimal solution, and establish the necessary and sufficient conditions of optimality under the assumption that the manifold of admissible loads is a closed convex set in some Hilbert space.

1. Certain relationships of the theory of plates. Auxiliary assumptions. We shall consider the inverse problems of plates of variable thickness. The equation of flexure of such a plate has the form [1]

$$Pu = \frac{\partial^2}{\partial x^2}\left[D\left(\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2}\right)\right] + \frac{\partial^2}{\partial y^2}\left[D\left(\frac{\partial^2 u}{\partial y^2} + \nu\frac{\partial^2 u}{\partial x^2}\right)\right] + 2(1-\nu)\frac{\partial^2}{\partial x\,\partial y}\left(D\frac{\partial^2 u}{\partial x\,\partial y}\right) = g, \quad (x, y) \in \Omega \tag{1.1}$$

Here $u(x, y)$ denotes the deflection of the median plane of the plate, ν is the Poisson's ratio which is a nonnegative constant, $g(x, y)$ is the external load intensity, $D(x, y)$ is the torsional rigidity of the plate and Ω is an open bounded region on the plane with the boundary S.

Let us consider the case of mixed boundary conditions when the plate is clamped along a part S_1 of the boundary, and freely supported along the remaining part S_2 of the boundary. The boundary conditions have the form

$$u|_S = 0, \quad \frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y = 0 \quad \text{on } S_1 \tag{1.2}$$

$$\nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + (1-\nu)\left(\frac{\partial^2 u}{\partial x^2} n_x^2 + \frac{\partial^2 u}{\partial y^2} n_y^2 + 2\frac{\partial^2 u}{\partial x \partial y} n_x n_y\right) = 0 \quad \text{on } S_2$$

Here n_x and n_y are the direction cosines of the outer unit vector normal to the boundary S ; $S = S_1 \cup S_2$ and $S_1 \cap S_2$ is an empty set. In particular, if one of the sets S_1 (S_2) is empty, then we have the case of a plate freely supported (clamped) along its whole edge.

We introduce now the Sobolev space $H^m(\Omega)$ ($m \geqslant 1$ is an integer)

$$H^m(\Omega) = \{v \mid D^\alpha v \in L_2(\Omega), \ |\alpha| \leqslant m\}$$

$$D^\alpha = \frac{\partial^{\alpha_1+\alpha_2}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}, \quad \alpha = (\alpha_1, \alpha_2), \quad |\alpha| = \alpha_1 + \alpha_2$$

The space $H^m(\Omega)$ has the norm

$$\|v\|_{H^m(\Omega)} = \Big(\sum_{|\alpha| \leqslant m} \|D^\alpha v\|^2_{L_2(\Omega)}\Big)^{1/2}$$

Let us denote by V a Hilbert space obtained by closure on the norm of the space $H^2(\Omega)$ of the set of smooth functions satisfying the first two conditions of (1.2), and introduce in the space V the following bilinear form:

$$a(u, v) = \iint_\Omega D\Big[\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) -$$
$$(1-\nu)\left(\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y}\right)\Big] dx\, dy$$

In what follows, we shall need the following assumptions:

1) $D(x, y)$ is a function measurable on Ω and satisfying the inequality

$$k_1 \leqslant D(x, y) \leqslant k_2, \quad k_1, \quad k_2 = \text{const} > 0 \tag{1.3}$$

almost everywhere on Ω.

2) The constant ν satisfies the inequality

$$0 \leqslant \nu < 1 \tag{1.4}$$

We shall show that in the space V the norm of the space $H^2(\Omega)$ is equivalent to the norm

$$\|u\|_V = [a(u, u)]^{1/2} \tag{1.5}$$

We have

$$\left|\iint_\Omega D \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} dx\, dy\right| \leqslant \frac{1}{2}\Big[\iint_\Omega D\left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx\, dy + \tag{1.6}$$
$$\iint_\Omega D\left(\frac{\partial^2 u}{\partial y^2}\right)^2 dx\, dy\Big]$$

Taking into account (1.4) and the inequalities (1.3) and (1.6), we obtain

$$a(u,u)=\iint_{\Omega} D\Big[\Big(\frac{\partial^2 u}{\partial x^2}\Big)^2+\Big(\frac{\partial^2 u}{\partial y^2}\Big)^2+2\nu\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}+ \tag{1.7}$$

$$2(1-\nu)\Big(\frac{\partial^2 u}{\partial x\,\partial y}\Big)^2\Big]dx\,dy \geqslant k_1(1-\nu)\iint_{\Omega}\Big[\Big(\frac{\partial^2 u}{\partial x^2}\Big)^2+$$

$$\Big(\frac{\partial^2 u}{\partial y^2}\Big)^2+2\Big(\frac{\partial^2 u}{\partial x\,\partial y}\Big)^2\Big]dx\,dy \geqslant c_1\iint_{\Omega}\Big[\Big(\frac{\partial u}{\partial x}\Big)^2+$$

$$\Big(\frac{\partial u}{\partial y}\Big)^2+u^2\Big]dx\,dy,\quad \forall u\in V,\quad c_1=\text{const}>0$$

The last estimate is obtained with the use of the Poincaré-Friedrichs inequality [2].From the inequalities (1.7) we obtain

$$a(u,\ u)\geqslant c_2\,\|u\|^2_{H_2(\Omega)},\quad \forall u\in V,\quad c_2=\text{const}>0$$

By virtue of the inequality (1.3) the converse inequality also holds

$$a(u,\ u)\leqslant c_3\,\|u\|^2_{H^2(\Omega)},\quad \forall u\in V,\ \ c_3=\text{const}>0$$

Let us introduce the following symmetric bilinear forms for the elements $u,\ v\in V$:

$$b(u,v)=\iint_{\Omega}\Big[(1+\nu^2)\Big(\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2}\Big)+ \tag{1.8}$$

$$+2\nu\Big(\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2}\Big)+\frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial^2 v}{\partial x\,\partial y}\Big]dx\,dy$$

Similarly to the previous arguments we can use the inequality (1.6) with $D=1$ to show, that the norm (1.5) in the space V is equivalent to the norm

$$\|u\|_1=[\,b(u,\ u)]^{1/2} \tag{1.9}$$

Thus the following assertion holds:

Lemma 1.1. The norm of the space $H^2(\Omega)$ is equivalent to the norms (1.5) and (1.9) in the assumptions (1) and (2) in the space V.

Let U be a Hilbert space on the field of real numbers and let U_∂ be a closed convex set in U.

Theorem 1.1. Let $\pi(f,\ g)$ be a continuous bilinear form symmetric on U, $\pi(f,\ g)=\pi(g,\ f)$, satisfying the condition

$$\pi(f,\ f)\geqslant C\,\|f\|_U{}^2,\quad \forall f\in U,\quad C=\text{const}>0 \tag{1.10}$$

and $L(f)$ a linear form continuous on U. Then a unique element $f_0\in U_\partial$ exists for which

$$\pi(f_0,f_0)-2L(f_0)=\inf_{g\in U_\partial}(\pi(g,g)-2L(g)) \tag{1.11}$$

The minimizing element f_0 is characterized by the fact that

$$\pi(f_0,\ g-f_0)-L(g-f_0)\geqslant 0,\quad \forall g\in U_\partial$$

If the bilinear symmetric form $\pi(f,\ g)$ continuous on U does not satisfy the condition (1.10) but the set U_∂ is bounded, then an element $f_0\in U_\partial$ exists which satisfies the condition (1.11).

A proof of this statement is given in [3]. The form $\pi(f,\ g)$ which satisfies the condition (1.10), is called coercive form.

2. Inverse problem for a plate with a special function for the bending moments. We shall call the generalized solution of the problem (1. 1), (1. 2) the function $u \in V$ for which the condition

$$a(u, h) = \iint_\Omega gh\, dx\, dy, \quad \forall h \in V \tag{2. 1}$$

holds.

From the results (see [4]) it follows:

Theorem 2. 1. Let the assumptions (1) and (2) hold and $g \in V^*$. Then the problem (1. 1), (1. 2) has a unique generalized solution u for which the following relation holds:

$$\| u \|_V = \| g \|_{V^*} \tag{2. 2}$$

Here and henceforth an asterisk denotes a conjugate space. Clearly the solution u of the problem (2. 1) depends on g. Equation (2. 1) defines this relationship unambiguously. Let us formulate the following problem: to find the load under which the distribution of the bending moments M_x and M_y and the torsional moment M_{xy} defined by the expressions

$$M_x = D\left(\frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^2 u}{\partial y^2}\right), \quad M_y = \left(\frac{\partial^2 u}{\partial y^2} + \nu \frac{\partial^2 u}{\partial x^2}\right), M_{xy} = D(1-\nu)\frac{\partial^2 u}{\partial x\, \partial y}$$

will be as near as possible, in the sense defined below, to their prescribed values.

We can express it more accurately by stating that, taking into account the assumptions (1) and (2), we consider the problem of a minimum of the functional

$$I(g) = \iint_\Omega \left[\left(\frac{\partial^2 u(g)}{\partial x^2} + \nu \frac{\partial^2 u(g)}{\partial y^2} - z_1\right)^2 + \left(\frac{\partial^2 u(g)}{\partial y^2} + \nu \frac{\partial^2 u(g)}{\partial x^2} - z_2\right)^2 + \left(\frac{\partial^2 u(g)}{\partial x\, \partial y} - z_3\right)^2\right] dx\, dy, \quad g \in V_\partial^* \tag{2. 3}$$

where V_∂^* is a closed convex set in space V^*, and $z = (z_1, z_2, z_3)$ is a specified element of the space $H = L_2(\Omega) \times L_2(\Omega) \times L_2(\Omega)$.

We introduce in the space V the following linear form:

$$Q_z(u) = \iint_\Omega \left[z_1\left(\frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^2 u}{\partial y^2}\right) + z_2\left(\frac{\partial^2 u}{\partial y^2} + \nu \frac{\partial^2 u}{\partial x^2}\right) + z_3 \frac{\partial^2 u}{\partial x\, \partial y}\right] dx\, dy, \quad u \in V \tag{2. 4}$$

and define on V^* the following forms:

$$\pi_1(f, g) = b(u(f), u(g)), \quad L_z(g) = Q_z(u(g)) \tag{2. 5}$$

where $b(\ .,\ .)$ is defined by the relation (1. 8).

Removing the brackets from the integrand in (2. 3), we find that the functional (2. 3) differs from the functional

$$I_1(g) = \pi_1(g, g) - 2L_z(g), \quad g \in V_\partial^* \tag{2. 6}$$

by a constant term.

Theorem 2. 2. Let the assumptions (1) and (2) hold, $z \in H$ and the function u be defined as the solution of the problem (2. 1). Then there exists a unique element $g_0 \in V_\partial^*$ for which

$$I(g_0) = \inf_{g \in V_\partial^*} I(g) \tag{2. 7}$$

The element g_0 is described by the relation (2. 1) in which $g = g_0$, and by the inequality

$$\pi_1(g_0, f - g_0) - L_z(f - g_0) \geqslant 0, \quad \forall f \in V_\partial^* \tag{2.8}$$

Proof. Using Theorem 1. 1 we can show that there exists a unique element $g_0 \in V_\partial^*$ for which

$$I_1(g_0) = \inf_{g \in V_\partial^*} I_1(g) \tag{2.9}$$

In fact, $\pi_1(f, g)$ is a bilinear and symmetric form. Using the first relation of (2. 5), Lemma 1. 1 and Theorem 2. 1, we obtain

$$\pi_1(g, g) = b(u(g), u(g)) \geqslant C_1 \| u(g) \|_V^2 = C_1 \| g \|_{V_*}^2, \ \forall g \in V^*$$
$$| \pi_1(f, g) | = |b(u(f), u(g))| \leqslant \| u(f) \|_1 \| u(g) \|_1 \leqslant C_2 \|f\|_{V_*} \|g\|_{V^*} \quad \forall f, g \in V^*$$

where C_1 and C_2 are positive constants. From this it follows that $\pi_1(f, g)$ is a coercive form continuous in V^*. From the relations (2. 2) and (2. 4) and the second relation of (2. 5), follows

$$| L_z(g) | \leqslant C_3 \| z \|_H \| u(g) \|_V = C_3 \| z \|_H \| g \|_{V^*}$$
$$\| z \|_H = \| z_1 \|_{L_2(\Omega)} + \| z_2 \|_{L_2(\Omega)} + \| z_3 \|_{L_2(\Omega)}, \quad C_3 = \text{const} > 0$$

therefore $L_z(g)$ is a linear form continuous on V^*.

Thus the conditions allowing for the application of Theorem 1. 1 hold, and a unique element $g_0 \in V_\partial^*$ exists for which the relations (2. 8) and (2. 9) hold. The functionals (2. 3) and (2. 6) differ from each other by a constant term, and the theorem is proved.

Let us transform the inequality (2. 8) using the conjugate state $p(g_0) \in V$ defined by the equation

$$a(p(g_0), h) = b(u(g_0), h) - Q_z(h), \quad \forall h \in V \tag{2.10}$$

Setting in (2. 10) $h = u(f) - u(g_0)$ and taking into account the relations (2. 5) and (2. 8), we obtain

$$\begin{aligned} &b(u(g_0), u(f) - u(g_0)) - Q_z(u(f) - u(g_0)) = \\ &a(p(g_0), u(f) - u(g_0)) = \iint_\Omega p(g_0)(f - g_0)\,dx\,dy \geqslant 0, \quad \forall f \in V_\partial^* \end{aligned} \tag{2.11}$$

Conversely, from the inequality

$$\iint_\Omega p(g_0)(f - g)\,dx\,dy \geqslant 0, \quad \forall f \in V_\partial^* \tag{2.12}$$

and relations (2. 10) follows (2. 8), and we have the following:

Corollary. Let the assumptions (1) and (2) hold and $z \in H$. The necessary and sufficient condition for the element $g_0 \in V_\partial^*$ to satisfy (2. 7) is, that the conditions (2. 1) hold for $g = g_0$, (2. 10) and (2. 12).

3. Inverse problem with a special function for the deflection. We formulate the problem of determining a load for which the deflection function will deviate as little as possible, in the sense defined below, from the given function. To express it more accurately, we consider the problem of a minimum of the following functional:

$$\Phi(g) = \iint_\Omega (u(g) - z)^2\,dx\,dy, \quad g \in V_\partial^* \tag{3.1}$$

where z is a specified element of the space $L_2(\Omega)$ and $u(g)$ is a solution of (2. 1).

In order to apply Theorem 1. 1, we introduce the following symmetric bilinear form:

$$\pi_2(f, g) = \iint_\Omega u(f)\, u(g)\, dx\, dy \qquad (3.2)$$

Taking into account (2. 2), we have

$$|\pi_2(f, g)| \leqslant \|u(f)\|_{L_2(\Omega)} \|u(g)\|_{L_2(\Omega)} \leqslant C \|u(f)\|_V \|u(g)\|_V = C\|f\|_{V^*}\|g\|_{V^*}, \quad C = \text{const} > 0$$

Therefore $\pi_2(f, g)$ is a form continuous on V^* It is not however a coercive form. Taking into account the fact that $z \in L_2(\Omega)$ and using (2. 2), we can confirm that the linear form

$$L_z(g) = \iint_\Omega u(g)\, z\, dx\, dy \qquad (3.3)$$

is continuous in V^*.

Let us now denote by X the set of such elements $f \in V_\partial^*$ that

$$\Phi(f) = \inf_{g \in V_\partial^*} \Phi(g) \qquad (3.4)$$

The form (3. 2) is not coercive, therefore the set X can be empty. However if the set V_∂^* is bounded, then X is not empty (this follows from Theorem 1. 1). If we replace the functional (3. 1) by the "regularized" functional

$$\Phi_1(g) = \Phi(g) + \alpha \|g\|^2_{V^*}, \quad \alpha > 0 \qquad (3.5)$$

then the corresponding bilinear form

$$\pi_3(f, g) = \iint_\Omega u(f)\, u(g)\, dx\, dy + \alpha [f, g]_{V^*}$$

where $[f, g]_{V^*}$ is a scalar product in V^*, will be coercive since $\pi_3(g, g) \geqslant \alpha \|g\|^2_{V^*}$. Applying Theorem 1. 1, we obtain the following:

Theorem 3. 1. Let the assumptions (1) and (2) hold, $z \in L_2(\Omega)$ and the function u be defined as the solution of the problem (2. 1). Then there exists a unique element $f \in V_\partial^*$ for which

$$\Phi_1(f) = \inf_{g \in V_\partial^*} \Phi_1(g) \qquad (3.6)$$

The element f is defined by the relation (2. 1) for $g = f$, and by the inequality

$$\iint_\Omega [u(f) - z][u(g) - u(f)]\, dx\, dy + \alpha [f, g - f]_{V^*} \geqslant 0, \quad \forall g \in V_\partial^* \qquad (3.7)$$

Let us define the conjugate condition $p(f) \in V$ as a solution of the equation

$$a(p(f), h) = \iint_\Omega (u(f) - z)\, h\, dx\, dy, \quad \forall h \in V \qquad (3.8)$$

Then, similarly to the previous case, we obtain the following:

Corollary. Let the assumptions (1) and (2) hold and $z \in L_2(\Omega)$. The necessary and sufficient condition for the element $f \in V_\partial^*$ to be a solution of the problem (3.6) is, that the relation (2. 1) for $g = f$ and the relation (3. 8) hold, as well as the inequality

$$\iint_\Omega p(f)(g - f)\, dx\, dy + \alpha [f, g - f]_{V^*} \geqslant 0, \quad \forall g \in V_\partial^*$$

Notes. 1. Let V_∂^* be a closed convex set in a finite-dimensional subspace V_n^* of the space V^*. In this case a unique element $f \in V_\partial^*$ exists for the functional (3. 1) such, that

$$\Phi(f) = \inf_{g \in V_\partial^*} \Phi(g) \tag{3.9}$$

Indeed, in a finite-dimensional space any two norms are equivalent, therefore

$$\| u(g) \|_{L_2(\Omega)} \geqslant C_4 \| u(g) \|_V, \quad \forall g \in V_n^*, \quad C_4 = \text{const} > 0 \tag{3.10}$$

Using the relations (2. 2), (3. 2) and (3. 10), we obtain

$$\pi_2(g, g) = \| u(g) \|^2_{L_2(\Omega)} \geqslant C_4^2 \| u(g) \|_V^2 = C_4^2 \| g \|^2_{V^*}, \quad \forall g \in V_n^*$$

Consequently the form $\pi_2(f, g)$ is coercive in V_n^* and the existence of a unique element $f \in V_\partial^*$ satisfying the relation (3. 9) follows from Theorem 1. 1.

2. We may consider the problem of a minimum of the functional

$$\Phi_2(g) = \Phi(g) + \alpha \| g \|^2_{L_2(\Omega)}, \quad \alpha > 0, \quad g \in L_{2\partial}(\Omega) \tag{3.11}$$

where $L_{2\partial}(\Omega)$ is a closed convex set in the space $L_2(\Omega)$.

Using similar reasoning we can show that there exists a unique element $f \in L_{2\partial}(\Omega)$ for which

$$\Phi_2(f) = \inf_{g \in L_{2\partial}(\Omega)} \Phi_2(g)$$

We note that when the set X of solutions of the nonregularized problem (3. 4) is nonempty, then the solutions of the problem (3. 6) converge in V^*, as $\alpha \to 0$, to the element f_0 which is a projection of the zero element in V^* on the set X (see [3]).

We have established above the existence of a unique solution of the optimization problem with a special function for the displacements in the case of regularization of the latter. We shall now show that under certain assumptions a unique solution of a nonregularized problem can also exist.

Let us make, in addition to (1) and (2), the following further assumptions:

3) Ω is a bounded region on a plane with a boundary S, the latter is an infinitely differentiable curve. Locally, Ω is on one side of S, i.e. $\Omega \cup S$ is a manifold with an edge belonging to class C^∞ (see [5, 6]);

4) The function $D(x, y)$ is infinitely differentiable in the region $\bar{\Omega} = \Omega \cup S$.

Let us consider the case of a plate rigidly clamped along the whole contour. The boundary conditions have the form

$$u = 0 \text{ on } S, \quad \frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y = 0 \text{ on } S \tag{3.12}$$

We denote by V_1 the space obtained by closure on the norm of the space $H^4(\Omega)$ of the set of smooth functions $u(x, y)$ satisfying (3. 12). We define, as a weak solution of the problem (1. 1), (3. 12), the function $u \in L_2(\Omega)$ for which

$$\iint_\Omega uPh\, dx\, dy = \iint_\Omega gh\, dx\, dy, \quad \forall h \in V_1 \tag{3.13}$$

where the operator P is determined by (1. 1).

Below we shall utilize the following result: if the assumptions (1) — (4) all hold and $\varphi(x)$ is a given function belonging to the space $L_2(\Omega)$, then the solution of the problem

$$Ph = \varphi, \quad h = 0 \quad \text{on } S, \quad \frac{\partial h}{\partial x} n_x + \frac{\partial h}{\partial y} n_y = 0 \quad \text{on } S$$

belongs to the space $H^4(\Omega)$ (see [7]). In other words, P is an isomorphism from V_1 onto $L_2(\Omega)$ (assertion A). Therefore a unique element $u \in L_2(\Omega)$ **exists for** $\forall g \in V_1^*$

for which the identity (3. 13) holds.

Next we shall consider the problem of a minimum of the functional

$$\Phi(g) = \iint_\Omega (u(g) - z)^2\, dx\, dy, \quad g \in V_{1\partial}^* \tag{3.14}$$

where z is a specified element of the space $L_2(\Omega)$, $u(g)$ is a solution of (3. 13) and $V_{1\partial}^*$ is a closed convex set in the space V_1^*

Theorem 3.2. Let the assumptions (1) — (4) all hold and $z \in L_2(\Omega)$. Then there exists a unique element $f \in V_{1\partial}^*$ for which

$$\Phi(f) = \inf_{g \in V_{1\partial}^*} \Phi(g) \tag{3.15}$$

This element f is characterized by the relation (3. 13) in which $g = f$, and the inequality

$$\iint_\Omega (u(f) - z)(u(g) - u(f))\, dx\, dy \geqslant 0, \quad \forall g \in V_{1\partial}^* \tag{3.16}$$

Proof. We shall show that the symmetric bilinear form

$$\pi(f, g) = \iint_\Omega u(f)\, u(g)\, dx\, dy$$

corresponding to the functional (3. 14) is continuous and coercive in V_1^*. Since the assumptions (1) — (4) all hold, we find that by virtue of the arguments given above, the assertion A holds. From this, setting $Ph_0 = u$ in (3. 13), we have

$$\| u \|^2_{L_2(\Omega)} = \iint_\Omega g h_0\, dx\, dy \leqslant \| g \|_{V_1^*} \| h_0 \|_{V_1} \leqslant c \| g \|_{V_1^*} \| u \|_{L_2(\Omega)}, \quad c = \text{const} > 0$$

which yields

$$\| u \|_{L_2(\Omega)} \| \leqslant c \| g \|_{V_1^*} \tag{3.17}$$

The last inequality gives

$$| \pi(f, g) | = \left| \iint_\Omega u(f)\, u(g)\, dx\, dy \right| \leqslant \| u(f) \|_{L_2(\Omega)} \| u(g) \|_{L_2(\Omega)} \leqslant c^2 \| f \|_{V_1^*} \| g \|_{V_1^*}$$

Consequently the form $\pi(f, g)$ is continuous on V_1^*. Furthermore, from (3. 13) and the assertion A, we have

$$\left| \iint_\Omega gh\, dx\, dy \right| = \left| \iint_\Omega uPh\, dx\, dy \right| \leqslant \| u \|_{L_2(\Omega)} \| Ph \|_{L_2(\Omega)} \leqslant \tag{3.18}$$

$$\leqslant c_1 \| u \|_{L_2(\Omega)} \| h \|_{V_1}, \quad c_1 = \text{const} > 0$$

From (3. 18) we obtain $\| g \|_{V_1^*} \leqslant c_1 \| u \|_{L_2(\Omega)}$, and we now have

$$\pi(g, g) = \iint_\Omega u^2(g)\, dx\, dy \geqslant \frac{1}{c_1^2} \| g \|^2_{V_1^*}$$

i.e. the form $\pi(f, g)$ is coercive. Taking into account the fact that $z \in L_2(\Omega)$ and using the inequality (3. 17), we can show that the linear form

$$L_z(g) = \iint_\Omega u(g)\, z\, dx\, dy$$

corresponding to the functional (3. 14) is continuous on V_1^*. The existence of a unique element $f \in V_{1\partial}^*$ for which the relations (3. 15) and (3. 16) hold, now follows from

Theorem 1.1.

Let us define the conjugate state $p\ (f) \in V_1$ as the solution of the problem

$$Pp\ (f) = u\ (f) - z \qquad (3.19)$$

Then the element f is characterized by the relations (3.13) in which $g = f$, (3.19) and the inequality

$$\iint_\Omega p\,(f)\,(g - f)\,dx\,dy \geqslant 0, \quad \forall g \in V^*_{1\partial} \qquad (3.20)$$

Let us now pass to the case of a hinged plate. We denote by V_2 the space obtained by closure on the norm of $H^4\ (\Omega)$ of the set of smooth functions $u\ (x,\ y)$ satisfying the boundary conditions

$$u = 0 \quad \text{на } S, \quad \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + (1 - \nu)\left(\frac{\partial^2 u}{\partial x^2} n_x^2 + \frac{\partial^2 u}{\partial y^2} n_y^2 + 2\frac{\partial^2 u}{\partial x\,\partial y} n_x n_y\right) = 0 \quad \text{на } S \qquad (3.21)$$

The function $u \in L_2\ (\Omega)$ for which

$$\iint_\Omega uPh\,dx\,dy = \iint_\Omega gh\,dx\,dy, \quad \forall h \in V_2 \qquad (3.22)$$

we shall call a weak solution of the problem (1.1), (3.21). If the assumptions (1) — (4) all hold, then a unique solution of the problem (3.22) exists for $\forall g \in V_2^*$. Let us consider the problem of a minimum of the functional

$$\Phi_1(g) = \iint_\Omega (u(g) - z)^2\,dx\,dy, \quad g \in V^*_{2\partial}$$

where $u\ (g)$ is the solution of the problem (3.22) and $V_{2\partial}{}^*$ is the closed convex set in the space $V_2{}^*$.

Using arguments similar to those used previously, we prove

Theorem 3.3. Let the assumptions (1) — (4) all hold and $z \in L_2\ (\Omega)$. Then there exists a unique element $f \in V_{2\partial}{}^*$ for which

$$\Phi_1\,(f) = \inf_{g \in V^*_{2\partial}} \Phi_1\,(g)$$

This element f is characterized by the relation (3.22) with $g = f$ and the inequality

$$\iint_\Omega (u\,(f) - z)\,(u\,(g) - u\,(f))\,dx\,dy \geqslant 0, \quad \forall g \in V^*_{2\partial}$$

The necessary and sufficient conditions of optimality can be established using the conjugate conditions which are expressed by the relation (3.22) with $g = f$, Eq. (3.19) in which $u\ (f)$ is the solution of the problem (3.22) and $p\ (f) \in V_2$, and the inequality (3.20) which must hold for every g belonging to $V_{2\partial}{}^*$.

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AXISYMMETRIC PROBLEM FOR AN ELASTIC SPACE WITH A SPHERICAL CUT

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An axisymmetric problem of deformation of a space weakened by a spherical cut with the external forces or displacements given at the edges of the cut, is solved in quadratures. The state of stress is expressed in terms of the analytic functions of a complex variable. The holomorphic character of these functions is studied and the nonholomorphic terms separated. Explicit formulas for the stresses on a surface complementing the cut to a complete sphere, are given for the case of uniform extension at infinity. The erroneous character of a number of solutions obtained earlier, is indicated.

1. Let an elastic space be weakened by a slit which coincides with a part of a spherical surface of unit radius with its center at the coordinate origin. In the meridional section the slit coincides with the arc AMB (see Fig. 1).

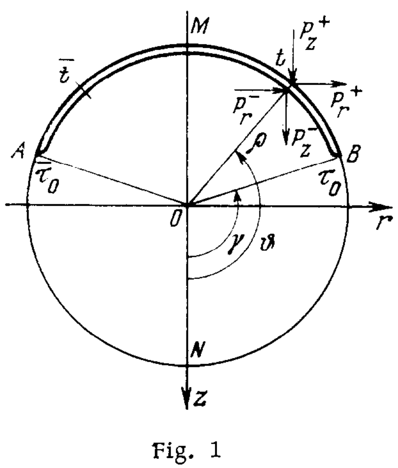

Fig. 1

The forces p_z^+, p_r^+ and p_z^-, p_r are given at the upper and lower edge of the slit, respectively. The stresses and displacements vanish at infinity. The displacments of the slit edges are assumed bounded, and although the stresses at these points may be infinite, their singularities must be of order strictly less than unity.

Similar assumptions were used in solving this problem in [1 — 4) and others, but the holomorphic character of the functions was wrongly assessed and the results obtained could therefore only be used for a restricted choice of loads.

The stresses in a body under an axisymmetric load are given in terms of two analytic functions φ and ψ of the complex variable ζ [5], by

$$\sigma_z = \frac{1}{\pi i}\int_{\bar{t}}^{t} (\varphi' - 2z\varphi'' - \psi')\, \frac{d\zeta}{X} \tag{1.1}$$